Chern-Simons Quantization of (2+1)-Anti-De Sitter Gravity on a Torus

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Abstract

Chern-Simons formulation of 2+1 dimensional Einstein gravity with a negative cosmological constant is investigated when the spacetime has the topology $\mathbf{R} \times T^2$. The physical phase space is shown to be a direct product of two sub-phase spaces each of which is a non-Hausdorff manifold plus a set with nonzero codimensions. Spacetime geometrical interpretation of each point in the phase space is also given and we explain the 1 to 2 correspondence with the ADM formalism from the geometrical viewpoint. In quantizing this theory, we construct a "modified phase space" which is a cotangnt bundle on a torus. We also provide a modular invariant inner product and investigate the relation to the quantum theory which is directly related to the spinor representation of the ADM formalism. (This paper is the revised version of a previous paper(hep-th/9312151). The wrong discussion on the topology of the phase space is corrected.)

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1 Introduction

Since the first order formalism of 2+1 dimensional Einstein gravity was shown to be equivalent to the Chern-Simons gauge theories with noncompact gauge groups[1][2], many works have appeared on this "Chern-Simons gravity"(CSG). Particularly in the case where the spacetime topology is $\mathbf{R} \times T^2$ and the cosmological constant vanishes, various aspects of CSG including its geometrical interpretation and the structure of its phase space seem to have been elucidated [3] [4] [5] [6] [7].

As for the case with nonvanishing cosmological constant, except a series of works on the holonomy algebra which are made by Nelson and Regge [8][9][10], relatively few people deal with this case[11]. In a previous paper[12], we have shown that the physical phase space in the negative cosmological constant case has nine sectors when the spacetime has the topology $\mathbf{R} \times T^2$ and one of these sectors is in 1 to 2 correspondence with the ADM phase space. However we knew little about the remaining eight sectors.

In this paper, we will give the topological and symplectic structures to the whole of this phase space. We find that this phase space is not equipped with a cotangent bundle structure, and that the topological structure is ditcinct according to whether we take the gauge group to be $\mathrm{SO}(2,1)_0 \times \mathrm{SO}(2,1)_0$ or $\widetilde{\mathrm{SL}}(2,\mathbf{R}) \times \widetilde{\mathrm{SL}}(2,\mathbf{R})$. We will give geometrical interpretations to each of the nine sectors in the $\mathrm{SO}(2,1)_0 \times \mathrm{SO}(2,1)_0$ case. Since the phase space does not have a cotangent bundle structure or a real polarization, we cannot naively apply the conventional quantization pocedure in which quantum states are represented by functions of "coordinates". To quantize such a phase space, we need in general the help of geometric quantization [13]. The discussion based on this geometric quantization, however, tends to be abstract. To make more concrete discussions be possible, we modify the phase space so that it can be a cotangent bundle. On the resulting phase space we can use the conventional procedure of canonical quantization.

In §2 we briefly review the Chern-Simoms formulation of anti-de Sitter gravity in the general case. We also explain how to reduce the phase space and how to obtain the symplectic structure of the reduced phase space. In §3, we investigate the phase space in the case where the spacetime is homeomorphic to $\mathbf{R} \times T^2$. We

give a new parametrization in terms of which the nine sectors which have already appeared in [12] can be described together. The relation of the new parameter with the other observables which were used in the previous works[8][9][10] [3][12] are shown in §4. §5 is devoted to the interpretation of the whole phase space in terms of the spacetime geometry. The 1 to 2 correspondence is also explained in the viewpoint of the spacetime geometry. In §6, we give a quantization prescription using the new parameres as basic variables. Transformation properties of the new variables under the modular group are also investigated both in the calssical and quantum theories. §7 is devoted to the discussion on the remaining issues.

Here we give the convention for the indices and the signatures of the metrics used in this paper:

1. $\mu, \nu, \rho, \cdots$ denote 2+1 dimensional spacetime indices and the metric $g_{\mu\nu}$ has the signature $(-, +, +)$.

2. $i, j, k, \cdots$ are used for spatial indices.

3. $a, b, c, \cdots$ represent indices of the $SO(2, 1)$ vector representation of the local Lorentz group, with the metric $\eta_{ab} = \text{diag}(-, +, +)$.

4. $\hat{a}, \hat{b}, \hat{c}, \cdots$ denote indices of the $SO(2, 2)$ vector representation of the anti-de Sitter group. The metric is given by $\eta_{\hat{a}\hat{b}} = \text{diag}(-, +, +, +)$.

5. ϵ_{abc} is the totally antisymmetric pseudo-tensor with $\epsilon_{012} = -\epsilon^{012} = 1$.

6. ϵ^{ij} denotes the totally antisymmetric tensor density on the spatial hypersurface Σ with $\epsilon^{12} = 1$.

2 Reduced Phase Space of Chern-Simons Gravity

We start with the first-order gravity in (2+1)-dimensions with a negative cosmological constant $\Lambda = -1/L^2$. We use as fundamental variables the triad $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$. The action is written as

$$
\begin{aligned}
I_{EP} &= \int_M \epsilon_{abc} e^a \wedge [d\omega^{bc} + \omega^b{}_d \wedge \omega^{dc} - \frac{1}{3}\Lambda e^b \wedge e^c] \\
&= L \int_M E^a (2d\omega_a + \epsilon_{abc}\omega^b\omega^c + \frac{1}{3}\epsilon_{abc}E^b E^c),
\end{aligned}
\tag{2.1}
$$

where we have introduced new variables $E^a \equiv \frac{1}{L}e^a$ and $\omega^a \equiv \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}$.

If we introduce the following (anti-)self-dual SO(2,1) connection

$$A^{(\pm)a} \equiv \omega^a \pm E^a,$$

the action (2.1) becomes the sum of two SO(2,1) Chern-Simons actions

$$
\begin{aligned}
I_{EP} &= \frac{L}{2}\int_M (\eta_{ab}A^{(+)a} \wedge dA^{(+)b} + \frac{1}{3}\epsilon_{abc}A^{(+)a} \wedge A^{(+)b} \wedge A^{(+)c}) \\
&\quad -\frac{L}{2}\int_M (\eta_{ab}A^{(-)a} \wedge dA^{(-)b} + \frac{1}{3}\epsilon_{abc}A^{(-)a} \wedge A^{(-)b} \wedge A^{(-)c})
\end{aligned}
\tag{2.2}
$$

up to surface terms. To proceed to the canonical formalism a la Witten, we assume that the spacetime M to be homeomorphic to $\mathbf{R} \times \Sigma$, where Σ is a two dimensional manifold [1], and we naively set $x^0 = t$

$$
\begin{aligned}
I_W &= (I_{EP})_{|M \approx R \times \Sigma} \\
&= \int dt \int_\Sigma d^2x (-\frac{L}{2}\epsilon^{ij}\eta_{ab}A_i^{(+)a}\dot{A}_j^{(+)b} + A_t^{(+)a}\mathcal{G}_a^{(+)}) \\
&\quad - \int dt \int_\Sigma d^2x[(+) \leftrightarrow (-)].
\end{aligned}
\tag{2.3}
$$

As is well known this is a first class constraint system. We have two kinds of first class constraints. First,

$$\Pi_{ta}^{(\pm)} \approx 0, \tag{2.4}$$

where $\Pi_{ta}^{(\pm)}$ is the momentum conjugate to $A_t^{(\pm)a}$. Second,

$$\mathcal{G}_a^{(\pm)} \equiv \frac{L}{2}\eta_{ab}\epsilon^{ij}(\partial_i A_j^{(\pm)b} - \partial_j A_i^{(\pm)b} + \epsilon^b{}_{cd}A_i^{(\pm)c}A_j^{(\pm)d}) \approx 0, \tag{2.5}$$

which are called as Gauss law constraints.

The phase space before the reduction is parametrized by $(A_i^{(\pm)a}, A_t^{(\pm)a}, \Pi_{ta}^{(\pm)})$, whose nonvanishing Poisson brackets can be read off from the action (2.3):

$$
\begin{aligned}
\{A_t^{(\pm)a}(x), \Pi_{tb}^{(\pm)}(y)\}_{P.B.} &= \pm\delta_b^a \delta^2(x,y), \\
\{A_i^{(\pm)a}(x), A_j^{(\pm)b}(y)\}_{P.B.} &= \pm\frac{1}{L}\eta^{ab}\epsilon_{ij}\delta^2(x,y).
\end{aligned}
\tag{2.6}
$$

---

[1]To simplify the analysis, we assume that Σ is compact and has no boundary.

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these are encoded in the symplectic structure of the unreduced system:

$$
\begin{aligned}
\Omega &= \Omega^{(+)} + \Omega^{(-)}, \\
\Omega^{(\pm)} &= \pm \int d^2x (-\frac{L}{2}\eta_{ab}\epsilon^{ij}\delta A_i^{(\pm)a} \wedge \delta A_j^{(\pm)b} + \delta\Pi_{ta}^{(\pm)} \wedge \delta A_t^{(\pm)a}),
\end{aligned}
\qquad (2.7)
$$

where δ denotes "exterior derivative on the phase space".

We will quantize the theory following the "reduced phase space method". Namely, we first solve the constraints to obtain the physical phase space, and then we consider the quantization on the physical phase space.

The first class constraints (2.4) and (2.5) tell us that the momentum $\Pi_{ta}^{(\pm)}$ conjugate to $A_t^{(\pm)a}$ vanishes and that $A_i^{(\pm)a}$ be flat $SO(2,1)_0$ connection on Σ. [2] To obtain the physical phase space, we further have to take the quotient space modulo gauge transformations which are generated by the first class constraints.

In our case, the generating functional of gauge transformation is

$$
G^{(\pm)}(N,\theta) \equiv \pm \int d^2x (N^a(x)\Pi_{ta}^{(\pm)}(x) + \theta_a(x)\mathcal{G}^{(\pm)a}(x)), \qquad (2.8)
$$

where N^a and θ^a in general depend on the dynamical variables. The infinitesimal transformation generated by (2.8) is (up to terms proportional to constraints which vanishes on the constraint surface where the constraint equations hold):

$$
\begin{aligned}
\delta_G A_i^{(\pm)a} &= \{A_i^{(\pm)a}, G^{(\pm)}(N,\theta)\}_{P.B.} = -D_i^{(\pm)}\theta^a \equiv -(\partial_i\theta^a + \epsilon^a{}_{bc}A_i^{(\pm)b}\theta^c), \\
\delta_G A_t^{(\pm)a} &= \{A_t^{(\pm)a}, G^{(\pm)}(N,\theta)\}_{P.B.} = N^a, \\
\delta_G \Pi_{ta}^{(\pm)} &= \{\Pi_{ta}^{(\pm)}, G^{(\pm)}(N,\theta)\}_{P.B.} = 0,
\end{aligned}
\qquad (2.9)
$$

i.e., the $SO(2,1)_0$ gauge transformation on $A_i^{(\pm)a}$ and the shift on $A_t^{(\pm)a}$. In principle, $A_t^{(\pm)a}$ can be arbitrarily chosen and we usually regard it as a Lagrange multiplier. Now the resulting phase space turns out to be a direct product $\mathcal{M}$ of two moduli spaces $\mathcal{M}^{(\pm)}$ of flat $SO(2,1)_0$ connections on Σ modulo $SO(2,1)_0$ gauge transformations:

$$
\mathcal{M} = \mathcal{M}^{(+)} \times \mathcal{M}^{(-)}. \qquad (2.10)
$$

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[2] In the gauge theory, we are often concerned with the identity component G_0 of the gauge group G.

Restriction of the symplectic sructure (2.7) to the constraint surface $\Pi_{ta}^{(\pm)} = \mathcal{G}^{(\pm)a} = 0$ naturally induces the symplectic structrue of the physical phase space $\mathcal{M}$. To see this, we compute the gauge transformation of (2.7):

$$
\begin{aligned}
\delta_G \Omega^{(\pm)} &= \pm \int d^2 x \left[-\frac{L}{2} \epsilon^{ij} \eta_{ab} \delta(-D_i^{(\pm)} \theta^a) \wedge \delta A_j^{(\pm)b} \times 2 + \delta \Pi_{ta}^{(\pm)} \wedge \delta N^a \right] \\
&= \int d^2 x [-\eta_{ab} \delta \theta^a \wedge \delta \mathcal{G}^{(\pm)b} + \delta \Pi_{ta}^{(\pm)} \wedge \delta N^a].
\end{aligned}
\tag{2.11}
$$

Because $\delta \mathcal{G}^{(\pm)a} = \delta \Pi_{ta}^{(\pm)} = 0$ on the constraint surface, we find $\delta_G \Omega = 0$ and we can regard the symplectic structure to be defined on $\mathcal{M}$.

Alternatively, we obtain the same result by properly fixing the gauge and by taking the Dirac bracket. A gauge-fixing corresponds to taking a "cross-section" which intersects with each orbit of the gauge transformations once and only once, and whose intersection with the constraint surface is isomorphic to the physical phase space. Dirac bracket is given by the symplectic structure of the "cross section" which is induced from the symplectic structure of the original unconstrained system. Taking these facts into account, we see that the symplectic structure of the physical phase space, i.e. eq.(2.7) restricted on $\mathcal{M}$, should be equivalent to the symplectic structure which is obtained by the Dirac bracket.

To parametrize $\mathcal{M}$, it is convenient to use holonomy of the connection $A^{(\pm)}$ [8][9]:

$$
h_A^{(\pm)}(\gamma) \equiv \mathcal{P} \exp \{ \int_1^0 ds \dot{\gamma}^i(s) A_i^{(\pm)}(\gamma(s)) \},
\tag{2.12}
$$

where $\gamma : [0,1] \to \Sigma$ is an arbitrary closed curve on Σ and the base point $x_0 = \gamma(0) = \gamma(1)$ is assumed to be fixed. $\mathcal{P}$ denotes the path ordered product, with larger s to the left.

Let us consider expressing the phase space $\mathcal{M}$ in terms of (2.12). Because the connection $A^{(\pm)}$ in $\mathcal{M}^{(\pm)}$ is flat, the $h_A^{(\pm)}$ depends only on the homotopy class of the closed curve γ. A gauge transformation of $A^{(\pm)}$

$$
A_i^{(\pm)}(x) \to A_i'^{(\pm)}(x) = g^{(\pm)}(x) A_i^{(\pm)}(x) g^{(\pm)-1}(x) - \partial_i g^{(\pm)}(x) g^{(\pm)-1}(x), \quad g^{(\pm)}(x) \in \mathrm{SO}(2,1)_0
$$

induces a conjugate transformation of $h_A^{(\pm)}$:

$$
h_A^{(\pm)} \to h_{A'}^{(\pm)} = g^{(\pm)}(x_0) h_A^{(\pm)} g^{(\pm)-1}(x_0).
$$

Hence we can express the physical phase space as

$$\mathcal{M}^{(\pm)} = Hom(\pi_1(\Sigma), SO(2,1)_0)/ \sim, \tag{2.13}$$

where $Hom(A, B)$ denotes the space of group homomorphisms $A \to B$, $\pi_1(\Sigma)$ is the fundamental group of Σ, and $\sim$ means the equivalence under the $SO(2,1)_0$ conjugations.

3 Reduced Phase Space on $\mathbf{R} \times T^2$

Now we apply the method explained in the last section to the case where $M \approx \mathbf{R} \times T^2$. First we look into the topological structure of the physical phase space.

The fundamental group $\pi_1(T^2)$ of a torus is generated by two commuting generators α and β. The holonomies of the flat connection $A^{(\pm)}$ therefore form a subgroup of $SO(2,1)_0$ generated by two commuting $SO(2,1)_0$ elements. By taking an appropriate conjugation, we know that each sub-phase spaces $\mathcal{M}^{(\pm)}$ consists of three subsectors $\mathcal{M}_S^{(\pm)}$, $\mathcal{M}_N^{(\pm)}$, and $\mathcal{M}_T^{(\pm)}$ [12] (plus a set $\mathcal{M}_0 = \{S^{(\pm)}[\alpha] = S^{(\pm)}[\beta] = 0\}$ with nonzero codimensions). [3] $\mathcal{M}_S^{(\pm)}$ is parametrized by

$$S^{(\pm)}[\alpha] = \exp(\lambda_2 \alpha_\pm), \quad S^{(\pm)}[\beta] = \exp(\lambda_2 \beta_\pm), \tag{3.1}$$

with $(\alpha_\pm, \beta_\pm) \in (\mathbf{R}^2 \backslash \{(0,0)\})/\mathbf{Z}_2$.[4] Parametrization of $\mathcal{M}_N^{(\pm)}$ is[5]

$$S^{(\pm)}[\alpha] = \exp\{(\lambda_0 \pm \lambda_2) \cos\theta_\pm\}, \quad S^{(\pm)}[\beta] = \exp\{(\lambda_0 \pm \lambda_2) \sin\theta_\pm\}, \tag{3.2}$$

with $\theta_\pm + 2\pi$ being identified with $\theta_\pm$. $\mathcal{M}_T^{(\pm)}$ is expressed by the following parametrization

$$S^{(\pm)}[\alpha] = \exp(\lambda_0 \rho_\pm), \quad S^{(\pm)}[\beta] = \exp(\lambda_0 \sigma_\pm), \tag{3.3}$$

where $\rho_\pm$ and $\sigma_\pm$ are periodic with period 2π.

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[3] We will use the spinor representation, where the generators of $SO(2,1)_0$ Lie algebra is given by pseudo-Pauli matrices λ_a:

$$\lambda_a \lambda_b = \frac{1}{4}\eta_{ab} + \frac{1}{2}\epsilon_{abc}\lambda^c.$$

We will henceforce denote the holonomy $h_A^{(\pm)}$ in the spinor representaion by $S^{(\pm)}$.

[4] $\mathbf{Z}_2$ in the denominator is generated by the internal inversion: $(\alpha_\pm, \beta_\pm) \to -(\alpha_\pm, \beta_\pm)$.

[5] This parametrization is different from that in [12]. In fact the former includes the latter as a special case with $\theta_\pm \in (-\pi/2, \pi/2)$.

To obtain their symplectic structures, we have to look for flat connections which give the desired holonomies. Such connections are easily found. If we use as coordinates on T^2 the periodic coordinates x and y along α and β with period 1, the simplest connections are the following

$$
\begin{aligned}
\mathcal{M}_S^{(\pm)} &: A^{(\pm)} \equiv A^{(\pm)a}\lambda_a = -\lambda_2(\alpha_\pm dx + \beta_\pm dy), \\
\mathcal{M}_N^{(\pm)} &: A^{(\pm)} = -(\lambda_0 + \lambda_2)(\cos\theta_\pm dx + \sin\theta_\pm dy), \\
\mathcal{M}_T^{(\pm)} &: A^{(\pm)} = -\lambda_2(\rho_\pm dx + \sigma_\pm dy).
\end{aligned}
\tag{3.4}
$$

The symplectic structures are obtained by substituting the above expressions for $A^{(\pm)}$ into eq.(2.7). The symplectic structure of $\mathcal{M}_S^{(\pm)}$ is given by

$$
\Omega^{(\pm)} = \mp L\delta\alpha_\pm \wedge \delta\beta_\pm.
\tag{3.5}
$$

$\mathcal{M}_N^{(\pm)}$ by itself does not have a symplectic structure. Symplectic structure of $\mathcal{M}_T^{(\pm)}$ is

$$
\Omega^{(\pm)} = \pm L\delta\rho_\pm \wedge \delta\sigma_\pm.
\tag{3.6}
$$

We would like to provide a construction in which these three subsectors $\mathcal{M}_S^{(\pm)}$, $\mathcal{M}_N^{(\pm)}$ and $\mathcal{M}_T^{(\pm)}$ appear in one parametrization. It turns out that this unification can be done as in the $\Lambda = 0$ case[6]. For this purpose we first consider two commuting $SO(2,1)_0$ holonomies in the following form:

$$
S^{(\pm)}[\alpha] = \exp\left[\cos\theta_\pm\left\{\left(r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_0 \pm \left(-r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_2\right\}\right],
$$

$$
S^{(\pm)}[\beta] = \exp\left[\sin\theta_\pm\left\{\left(r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_0 \pm \left(-r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_2\right\}\right]
\tag{3.7}
$$

The corresponding connection is given by

$$
A^{(\pm)} = -\left\{\left(r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_0 \pm \left(-r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\lambda_2\right\}(\cos\theta_\pm dx + \sin\theta_\pm dy).
\tag{3.8}
$$

The above connection with $r_\pm < 0$, $r_\pm = 0$ and $r_\pm > 0$ give parametrization of $\mathcal{M}_S$, $\mathcal{M}_N$ and $\mathcal{M}_T$ respectively. Relations between these new parameters $(r_\pm, \theta_\pm)$ and the old ones $(\alpha_\pm, \beta_\pm)$ for $\mathcal{M}_S^{(\pm)}$ and $(\rho_\pm, \sigma_\pm)$ for $\mathcal{M}_T^{(\pm)}$ are obtained by performing on (3.7) the conjugation using $\exp(\mp\lambda_1\Phi_\pm)$ with $\Phi_\pm = \frac{1}{2}\ln\{|r_\pm|/(\sqrt{r_\pm^2 + 1} + 1)\}$:

$$
(\alpha_\pm, \beta_\pm) = \pm\sqrt{-2r_\pm}(\cos\theta_\pm, \sin\theta_\pm) \quad for \quad r_\pm < 0,
\tag{3.9}
$$

$$
(\rho_\pm, \sigma_\pm) = \sqrt{2r_\pm}(\cos\theta_\pm, \sin\theta_\pm) \quad for \quad r_\pm > 0.
\tag{3.10}
$$

We should note that for $r_\pm > 0$, the parameters $(r_\pm, \theta_\pm)$ are subject to somewhat complicated identification conditions due to the periodicity of $(\rho_\pm, \sigma_\pm)$.

Using the new parametrization, symplectic structures (3.5) and (3.6) are expressed by the unified form:

$$\pm L\delta r_\pm \wedge \delta\theta_\pm. \tag{3.11}$$

In this expression, vanishing of the symplectic structure in $\mathcal{M}_N^{(\pm)}$ can be also explained by the fact that $r_\pm$ is a constant (*i.e.* zero) in this subsector.

In summary, we give the topological structure of

$$\mathcal{M}_U^{(\pm)} \equiv \mathcal{M}^{(\pm)} \backslash \mathcal{M}_0^{(\pm)} = \mathcal{M}_S^{(\pm)} \cup \mathcal{M}_N^{(\pm)} \cup \mathcal{M}_T^{(\pm)}.$$

We should notice that the period of the parameter $\theta_\pm$ is π for $r_\pm < 0$ and 2π for $r_\pm \geq 0$. The $\mathcal{M}_U^{(\pm)}$ defined above therefore turns out to be a non-Hausdorff manifold constructed by gluing together a punctured cone ($\mathcal{M}_S^{(\pm)}$) and a punctured torus ($\mathcal{M}_T^{(\pm)}$) at the puncture in the one to two fashion. The circle which serves as the glue is provided by $\mathcal{M}_N^{(\pm)}$. This structure precisely coincides with that of the base space of cotangent bundle structure of the phase space in the case with a vanishing cosmological constant [6]. In the case with a negative cosmological constant, however, the phase space $\mathcal{M}$ does not have a cotangent bundle structure even after the removal of the set involving $\mathcal{M}_0^{(\pm)}$. The phase space is represented by the direct product of two non-Hausdorff manifolds plus the set with nonzero codimensions.

Here we make a remark. In obtaining the sub-phase space $\mathcal{M}^{(\pm)}$, we first found out an adequate $SO(2,1)_0$ holonomy and then consructed the corresponding $SO(2,1)_0$ connection. In fact, this procedure involves identifying the connections which are related with each other by a large gauge transformation

$$g^{(\pm)} = \exp\{(2\pi\lambda_0(nx + my)\} \quad (n, m \in \mathbf{Z}). \tag{3.12}$$

Since $SO(2,1)_0$ (or $SL(2,\mathbf{R})$) is not simply connected, this class of gauge transformations cannot be generated by the first class constraints (2.8). Whether we should incorporate such a symmetry or not depends on physical considerations. If

we consider the symmetry under large gauge transformations (3.12) to be "physically irrelevant", the result is equivalent to that obtained when we use as a gauge group the universal covering group $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of $\mathrm{SO}(2,1)_0$. In that case the reduced phase space $\tilde{\mathcal{M}}$ is the direct product of two sub-phase spaces $\tilde{\mathcal{M}}^{(\pm)}$:

$$\tilde{\mathcal{M}}^{(\pm)} = \left(\bigcup_{n,m\in\mathbf{Z}} \mathcal{M}_S^{(\pm)nm}\right) \cup \left(\bigcup_{n,m\in\mathbf{Z}} \mathcal{M}_N^{(\pm)nm}\right) \cup \tilde{\mathcal{M}}_T^{(\pm)} \cup \left(\bigcup_{n,m\in\mathbf{Z}} \mathcal{M}_0^{(\pm)nm}\right).(3.13)$$

Connection which belongs to each subsector is [6]

$$\mathcal{M}_S^{(\pm)nm} \ : \ A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy)$$
$$-(\tilde{\lambda}_2 \cos 2\pi(nx+my) - \tilde{\lambda}_1 \sin 2\pi(nx+my))(\alpha_\pm dx + \beta_\pm dy)$$
$$\mathcal{M}_N^{(\pm)nm} \ : \ A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy)$$
$$-[\tilde{\lambda}_0 \pm (\tilde{\lambda}_2 \cos 2\pi(nx+my) - \tilde{\lambda}_1 \sin 2\pi(nx+my))](\cos\theta_\pm dx + \sin\theta_\pm dy)$$
$$\tilde{\mathcal{M}}_T^{(\pm)} \ : \ A^{(\pm)} = -\tilde{\lambda}_0(\tilde{\rho}_\pm dx + \tilde{\sigma}_\pm dy)$$
$$\mathcal{M}_0^{(\pm)nm} \ : \ A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy), \tag{3.14}$$

where the parameters $\alpha_\pm$, $\beta_\pm$ and $\theta_\pm$ run in the same regions as those in the $\mathrm{SO}(2,1)_0$ case, but the domain of $(\tilde{\rho}_\pm, \tilde{\sigma}_\pm)$ is $\mathbf{R}^2\backslash\{(2\pi n, 2\pi m)|n,m\in\mathbf{Z}\}$.

As in the $\mathrm{SO}(2,1)_0$ case we can "unify" the sub-phase space except the set $\bigcup_{n,m\in\mathbf{Z}} \mathcal{M}_0^{(\pm)nm}$ with nonzero codimensions. Though we cannot give coordinates which parametrize the whole of $\tilde{\mathcal{M}}_U^{(\pm)} = \tilde{\mathcal{M}}^{(\pm)}\backslash\left(\bigcup_{n,m\in\mathbf{Z}} \mathcal{M}_0^{(\pm)nm}\right)$, we can find a chart in the neighbourhood of $A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy) \in \mathcal{M}_0^{(\pm)nm}$:

$$A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy)$$
$$-\left\{\left(r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2}\tilde{\lambda}_0 \pm \left(-r_\pm + \sqrt{r_\pm^2 + 1}\right)^{1/2} \right. \tag{3.15}$$
$$\left. \times \left(\tilde{\lambda}_2 \cos 2\pi(nx+my) - \tilde{\lambda}_1 \sin 2\pi(nx+my)\right)\right\}(\cos\theta_\pm dx + \sin\theta_\pm dy).$$

The above connection with $r_\pm < 0$, $r_\pm = 0$ and $r_\pm > 0$ give parametrizations of $\mathcal{M}_S^{(\pm)nm}$, $\mathcal{M}_N^{(\pm)nm}$ and $\tilde{\mathcal{M}}_T^{(\pm)}$ respectively. Relations between the old and the new parameters are exactly the same as those in the $\mathrm{SO}(2,1)_0$ case, provided that $(\rho_\pm, \sigma_\pm)$ be replaced by $(\tilde{\rho}_\pm - 2\pi n, \tilde{\sigma}_\pm - 2\pi m)$.

Since we have obtained a chart, the topological structure of $\tilde{\mathcal{M}}_U^{(\pm)}$ can be read off. Local structure of the neighbourhood of $A^{(\pm)} = -\tilde{\lambda}_0 2\pi(ndx + mdy)$ precisely

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[6]The $\tilde{\lambda}_a$ is the generator of $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ and is subject to the same commutation relations as that of pseudo-Pauli matrices.

coinsides with that of the neighbourhood of $A^{(\pm)} = 0$ in the $SO(2,1)_0$ case. Globally, $\tilde{\mathcal{M}}_U^{(\pm)}$ is a non-Hausdorff manifold which is obtained by gluing infinitely many copies of a punctured cone $(\mathcal{M}_S^{(\pm)nm})$ to an infinitely many punctured plane $(\tilde{\mathcal{M}}_T^{(\pm)})$ at each puncture in the one-to-two fashion. $\mathcal{M}_N^{(\pm)nm}$ serve as the glue.

4 Relation to Other Formalisms

In the last section we have provided the new parametrization and investigated the topology of the phase space of CSG. Our choice of basic variables is, however, somewhat different from those of the previous literatures on CSG [8][9] [3][12]. Now we will make the relations between our variables and conventional ones transparent.

First we investigate relations to the invariants of Nelson and Regge [8][9]. Nelson and Regge used the Wilson loop operator in the chiral spinor representation to parametrize the physical phase space. Our (anti-)self-dual holonomy $S^{(\pm)}[\gamma]$ essentially corresponds to the "integrated connection" $S^\pm(\gamma)$ in [9], so we can easily express the c-invariants of Nelson and Regge in terms of our new variables

$$(c^\pm(\alpha), c^\pm(\beta)) = \begin{cases} (\cosh \frac{\alpha_\pm}{2}, \cosh \frac{\beta_\pm}{2}) & for \quad \mathcal{M}_S^{(\pm)} \\ (\quad 1 \quad, \quad 1 \quad) & for \quad \mathcal{M}_N^{(\pm)} \\ (\cos \frac{\rho_\pm}{2}, \cos \frac{\sigma_\pm}{2}) & for \quad \mathcal{M}_T^{(\pm)}. \end{cases} \tag{4.1}$$

These can formally be rewritten in a unified fashion

$$(c^\pm(\alpha), c^\pm(\beta)) = \left(\cos(\sqrt{\frac{r_\pm}{2}} \cos \theta_\pm), \cos(\sqrt{\frac{r_\pm}{2}} \sin \theta_\pm) \right). \tag{4.2}$$

Now we can give an alternative derivation of the Poisson bracket, or the symplectic structure (3.5)(3.6). The Poisson bracket of c-invariants is given in ref.[9]. After translating into our convention, it is

$$\{c^\pm(\alpha), c^\pm(\beta)\}_{P.B.} = \pm \frac{1}{8L} (c^\pm(\alpha\beta) - c^\pm(\alpha\beta^{-1})). \tag{4.3}$$

Substituting eq.(4.1) into eq.(4.3), we find, for example for $\mathcal{M}_S^{(\pm)}$

$$\{\cosh \frac{\alpha_\pm}{2}, \cosh \frac{\beta_\pm}{2}\}_{P.B.} = \pm \frac{1}{4L} \sinh \frac{\alpha_\pm}{2} \sinh \frac{\beta_\pm}{2},$$

which is equivalent to (3.5) classically. Similar calculation shows the equivalence of eq.(4.3) to eq.(3.6) and to eq.(3.11).

Next we consider the relation to the ADM formalism[14]. In a previous paper [12] we have investigated relations between the ADM formalism and CSG in detail when $\Lambda \neq 0$. We have shown that the ADM formalism has direct correspondence with the $\left(\mathcal{M}_S^{(+)} \times \mathcal{M}_S^{(-)}\right)$-sector of the physical phase space of CSG. We have also shown that the ADM variables (complex modulus m, conjugate momentum p and Hamiltonian H) can be expressed in terms of the parameters (α, β, u, v) which were used in [12] to parametrize $\mathcal{M}_S^{(+)} \times \mathcal{M}_S^{(+)}$:

$$
\begin{aligned}
m &= \frac{v + i\beta \tan t}{u + i\alpha \tan t} \\
p &= -iL \cot t (u - i\alpha \tan t)^2 \\
H &= -\frac{L}{\sin t \cos t}(u\beta - v\alpha).
\end{aligned}
\tag{4.4}
$$

The canonical transformation from the ADM variables to the (α, β, u, v)-variables is written as

$$
\mathrm{Re}(p\delta\overline{m}) - H\delta t = 2L(v\delta\alpha - u\delta\beta) + \delta F
\tag{4.5}
$$

where

$$
F(m_1, m_2, \alpha, \beta) = \frac{L \tan t}{m_2}|\beta - m\alpha|^2.
\tag{4.6}
$$

So it is sufficient to show the relation between our new paramtrization and the old one (α, β, u, v). By considering that these partameters are originally used to express holonomies, it is straightforward to find

$$
\alpha_\pm = \alpha \pm u \quad , \quad \beta_\pm = \beta \pm v.
\tag{4.7}
$$

Using (4.4) , (4.7) and (3.9), we find the expressions of the ADM variables in terms of new parameters $(r_\pm, \theta_\pm)$:

$$
m = \frac{e^{it} \sin\theta_+ \sqrt{-2r_+} + e^{-it} \sin\theta_- \sqrt{-2r_-}}{e^{it} \cos\theta_+ \sqrt{-2r_+} + e^{-it} \cos\theta_- \sqrt{-2r_-}} \quad ,
\tag{4.8}
$$

$$
p = \frac{-iL}{4 \sin t \cos t}\left(e^{-it} \cos\theta_+ \sqrt{-2r_+} + e^{it} \cos\theta_- \sqrt{-2r_-}\right)^2 ,
\tag{4.9}
$$

$$
H = \frac{-L}{\sin t \cos t} \sin(\theta_+ - \theta_-)\sqrt{r_+ r_-} \quad ,
\tag{4.10}
$$

which are essentially the same as those given in ref.[10]. These new parameters $(r_\pm, \theta_\pm)$ are related with the parameters (α, β, u, v) by an ordinary canonical transformation

$$
\begin{aligned}
2L(v\delta\alpha - u\delta\beta) &= L(r_+\delta\theta_+ - r_-\delta\theta_-) - \delta V, \\
V(\alpha, \beta, \theta_+, \theta_-) &= 2L\frac{(\alpha \sin\theta_- - \beta \cos\theta_-)(\alpha \sin\theta_+ - \beta \cos\theta_+)}{\sin(\theta_+ - \theta_-)}
\end{aligned} \quad . \quad (4.11)
$$

However, the canonical transformation from the ADM variables to these new parameters is *singular* in the sense that it does not contain the generating function:

$$
\mathrm{Re}(p\delta\overline{m}) - H\delta t = L(r_+\delta\theta_+ - r_-\delta\theta_-). \tag{4.12}
$$

We conjecture that this singular nature is related to the fact that $r_\pm$ and therefore p cannot be expressed in terms of $(m, \overline{m}, \theta_+, \theta_-)$ alone.

We know that the $\mathcal{M}_{(S,S)}$ is in 1 to 2 correspondence with the ADM formalism [12]. This is originated from the symmetry of CSG under the transformation

$$
(\alpha, \beta, u, v) \rightarrow (u, v, \alpha, \beta),
$$

which can be expressed in terms of the ADM formalism by

$$
t \rightarrow t + \frac{\pi}{2}.
$$

In the next section, we will look into this 1 to 2 correspondence from the viewpoint of the spacetime geometry.

5 Geometrical Interpretation of the Reduced Phase Space

In this section we try to relate a spacetime to each point in the physical phase space. We mainly focus on the case where the gauge group is $SO(2,1)_0 \times SO(2,1)_0$. We use (x, y) as periodic coordinates on T^2 with period 1. Identification conditions are therefore obvious. Since the set involving $\mathcal{M}_0^{(\pm)}$ gives singular universes, we only consider the subspace $\mathcal{M}' \equiv \mathcal{M}_U^{(+)} \times \mathcal{M}_U^{(-)}$ with codimension zero, which consists of the nine sectors. We will denote these sectors as $\mathcal{M}_{(\Psi,\Phi)} \equiv \mathcal{M}_\Psi^{(+)} \times \mathcal{M}_\Phi^{(-)}$ ($\Psi, \Phi = S, N, T$).

As an illustration we review the spacetime construction from $\mathcal{M}_{(S,S)}$ [12]. The simplest connection which gives the holonomies (3.1) is given by (3.4):

$$A^{(\pm)} = -\lambda_2 d\varphi_{\pm}, \quad (\varphi_{\pm} \equiv \alpha_{\pm}x + \beta_{\pm}y). \tag{5.1}$$

By performing a time-dependent gauge transformation $g^{(\pm)} = e^{\mp\lambda_0 t}$ and by extracting the triad part

$$E^0 = dt, \quad E^1 = -\sin t\frac{d\varphi_+ + d\varphi_-}{2}, \quad E^2 = -\cos t\frac{d\varphi_+ - d\varphi_-}{2}, \tag{5.2}$$

we can construct the spacetime metric

$$L^{-2}ds^2 = -dt^2 + \cos^2 t \; d\left(\frac{\varphi_+ - \varphi_-}{2}\right)^2 + \sin^2 t \; d\left(\frac{\varphi_+ + \varphi_-}{2}\right)^2. \tag{5.3}$$

Parametrization of the AdS^3 which reproduces this metric is:

$$(T, X, Y, Z) = L(\sin t \cosh \tfrac{\varphi_+ + \varphi_-}{2}, \quad \sin t \sinh \tfrac{\varphi_+ + \varphi_-}{2}, \\ \cos t \sinh \tfrac{\varphi_+ - \varphi_-}{2}, \cos t \cosh \tfrac{\varphi_+ - \varphi_-}{2}). \tag{5.4}$$

We should remark that the periodicity condition for the above parametrization is expressed by the identification under two $SO(2,2)_0$ transformations of $(T, X, Y, Z) \in M^{2+2}$

$$\tilde{E}[\alpha] = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & \cosh u & \sinh u \\ 0 & 0 & \sinh u & \cosh u \end{pmatrix}$$

$$\tilde{E}[\beta] = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{pmatrix}, \tag{5.5}$$

which are given by the (anti-)self-dual $SO(2,1)_0$ holonomies (3.1) through the relation

$$S^{(+)}\tilde{\sigma}_{\hat{a}}X^{\hat{a}}[S^{(-)}]^{-1} = \tilde{\sigma}_{\hat{a}}\tilde{E}^{\hat{a}}_{\hat{b}}X^{\hat{b}}, \tag{5.6}$$

where $\tilde{\sigma}_{\hat{a}} \equiv (2\lambda_a, \mathbf{1})$ is the "soldering form" in 2+2 dimensional Minkowskii space:

$$L^{-1}\tilde{\sigma}_{\hat{a}}X^{\hat{a}} = L^{-1}\begin{pmatrix} Y + Z & T + X \\ -T + X & -Y + Z \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{Z}). \tag{5.7}$$

The spacetime construction of Witten and Mess[1][15], in which we identify the spacetime M with a quotient space $\mathcal{F}/G$, where $\mathcal{F}$ is a subspace of the anti-de Sitter

space AdS^3 and G is a subgroup of $SO(2,2)$ which is specified by a point on the physical phase space, therefore seems to be equivalent to the standard construction explained above.

Indeed, it turns out that these two alternative constructions give the same spacetime also to the remaining eight sectors. We will omit the detail of its derivation and give only parametrization in the AdS^3 which represent the spacetime constructed from a point in each sectors.[7, 8]

$\mathcal{M}_{(N,N)}$:

$$X + Z = Le^t, \quad (T,Y) = Le^t \left(\tfrac{\eta_+ - \eta_-}{2}, \tfrac{\eta_+ + \eta_-}{2} \right). \tag{5.8}$$

$\mathcal{M}_{(T,T)}$:

$$L^{-1}(T,X,Y,Z) = (\cosh t \cos \tfrac{\zeta_+ - \zeta_-}{2}, \quad \sinh t \cos \tfrac{\zeta_+ + \zeta_-}{2}, \\ \sinh t \sin \tfrac{\zeta_+ + \zeta_-}{2}, -\cosh t \sin \tfrac{\zeta_+ - \zeta_-}{2}). \tag{5.9}$$

$\mathcal{M}_{(T,S)}$:

$$L^{-1}(T,X,Y,Z) = \\ \cosh t (\cosh \tfrac{\varphi_-}{2} \cos \tfrac{\zeta_+}{2}, \sinh \tfrac{\varphi_-}{2} \cos \tfrac{\zeta_+}{2}, \sinh \tfrac{\varphi_-}{2} \sin \tfrac{\zeta_+}{2}, -\cosh \tfrac{\varphi_-}{2} \sin \tfrac{\zeta_+}{2}) \\ + \sinh t (\sinh \tfrac{\varphi_-}{2} \sin \tfrac{\zeta_+}{2}, \cosh \tfrac{\varphi_-}{2} \sin \tfrac{\zeta_+}{2}, -\cosh \tfrac{\varphi_-}{2} \cos \tfrac{\zeta_+}{2}, \sinh \tfrac{\varphi_-}{2} \cos \tfrac{\zeta_+}{2}). \tag{5.10}$$

$\mathcal{M}_{(S,N)}$:

$$L^{-1}(T,Y) = (\sin t \cosh \tfrac{\varphi_+}{2}, \cos t \sinh \tfrac{\varphi_+}{2}) + \tfrac{\eta_-(\cos t \cosh \frac{\varphi_+}{2} + \sin t \sinh \frac{\varphi_+}{2})}{2}(-1,1),$$
$$L^{-1}(Z,X) = (\cos t \cosh \tfrac{\varphi_+}{2}, \sin t \sinh \tfrac{\varphi_+}{2}) - \tfrac{\eta_-(\cos t \sinh \frac{\varphi_+}{2} + \sin t \cosh \frac{\varphi_+}{2})}{2}(-1,1). \tag{5.11}$$

$\mathcal{M}_{(T,N)}$:

$$L^{-1}(T,Y) = (\cosh t \cos \tfrac{\zeta_+}{2}, \sinh t \sin \tfrac{\zeta_+}{2}) + \tfrac{\eta_-(\sinh t \cos \frac{\zeta_+}{2} - \cosh t \sin \frac{\zeta_+}{2})}{2}(-1,1),$$
$$L^{-1}(Z,X) = (-\cosh t \sin \tfrac{\zeta_+}{2}, \sinh t \cos \tfrac{\zeta_+}{2}) - \tfrac{\eta_-(\cosh t \cos \frac{\zeta_+}{2} + \sinh t \sin \frac{\zeta_+}{2})}{2}(-1,1). \tag{5.12}$$

---

[7] We always consider that
$$T^2 - X^2 - Y^2 + Z^2 = L^2$$
holds. The metric is obtained by substituting the parametrization into the pseudo-Minkowski metric:
$$ds^2 = -dT^2 + dX^2 + dY^2 - dZ^2.$$

[8] We define the following new coordinates on T^2:
$$\eta_\pm \equiv x \cos \theta_\pm + y \sin \theta_\pm, \quad \zeta_\pm \equiv \rho_\pm x + \sigma_\pm y.$$

As for the other three sectors $\mathcal{M}_{(S,T)}$, $\mathcal{M}_{(N,S)}$ and $\mathcal{M}_{(N,T)}$, the following holds generically. The metric obtained from a point in $\mathcal{M}_{(\Phi,\Psi)}$ ($\Phi \neq \Psi$) can be made into the same form as the one obtained from $\mathcal{M}_{(\Psi,\Phi)}$ with the subscripts $\pm$ replaced by $\mp$. On the other hand, the triad and the parametrization in the former are respectively obtained by reversing the orientation of the triad and by replacing Z with $-Z$ in the latter. This is also expected from the fact that interchanging $S^{(+)}$ and $S^{(-)}$ in eq.(5.6) is equivalent to the conjugation of $\tilde{E}$ by

$$P_Z : (T, X, Y, Z) \rightarrow (T, X, Y, -Z).$$

Taking these facts into account, we can say that the universe obtained from a point in $\mathcal{M}_{(\Phi,\Psi)}$ is the "mirror image" of that in $\mathcal{M}_{(\Psi,\Phi)}$.

Here we remark a few problems of the spacetime interpretation of this type. In the above discussion we have neglected whether the action of holonomy group is properly discontinuous. Let us consider $\mathcal{M}_{(T,T)}$ as an illustration. The $SO(2,2)_0$ holonomies in this sector is expressed by combining the rotations in the (X,Y)- and (T,Z)-directions. If we consider to take the quotient of the anti-de Sitter space, the action of the holonomy group is not properly discontinuous. To make the action of the holonomy properly discontinuous we have to i) take the universal covering $\widetilde{AdS}^3$ of the anti-de Sitter space and ii) remove $X = Y = 0(T^2 + Z^2 = L^2)$ from $\widetilde{AdS}^3$ and take the universal covering of the resultant space. After performing these prescriptions the quotient space is made well-defined. There is, however, another problem. To a point $(\rho_+, \sigma_+, \rho_-, \sigma_-)$ on $\mathcal{M}_{(T,T)}$, there correspond infinitely many spacetimes which are obtained by replacing $(\rho_\pm, \sigma_\pm)$ in the parametrization (5.9) by $(\rho_\pm + 2\pi m_\pm, \sigma_\pm + 2\pi n_\pm)$. In fact such situation is generic to the sectors $\mathcal{M}_{(T,\Psi)}$ and $\mathcal{M}_{(\Phi,T)}$. At first sight this problem seems to be settled down by considering the $\widetilde{SL}(2,\mathbf{R}) \times \widetilde{SL}(2,\mathbf{R})$ gauge theory. The problem is, however, not so simple because it is difficult to deal with $\mathcal{M}_S^{(\pm)nm}$ or $\mathcal{M}_N^{(\pm)nm}$. Consider $\mathcal{M}_S^{(+)n_+m_+} \times \mathcal{M}_S^{(+)n_-m_-}$ as an example. The original connection is given by (3.14). As in the case of $\mathcal{M}_{(S,S)}$, by performing the time-dependent gauge transformation $g^{(\pm)} = e^{\mp\lambda_0 t}$ and by extracting

the triad part,

$$
\begin{aligned}
E^0 &= dt - \pi\{(n_+ - n_-)dx + (m_+ - m_-)dy\} \equiv dt' \\
\begin{pmatrix} E^1 \\ E^2 \end{pmatrix} &= \begin{pmatrix} \cos\Theta(x,y) & -\sin\Theta(x,y) \\ \sin\Theta(x,y) & \cos\Theta(x,y) \end{pmatrix} \times \begin{pmatrix} -\sin t' \frac{d\varphi_+ + d\varphi_-}{2} \\ -\cos t' \frac{d\varphi_+ - d\varphi_-}{2} \end{pmatrix},
\end{aligned} \tag{5.13}
$$

where $\Theta(x,y) \equiv \pi\{(n_+ + n_-)x + (m_+ + m_-)y\}$, we can construct the spacetime metric

$$
ds^2 = L^2 \left(-dt'^2 + \sin^2 t' (\frac{d\varphi_+ + d\varphi_-}{2})^2 + \cos^2 t' (\frac{d\varphi_+ - d\varphi_-}{2})^2 \right), \tag{5.14}
$$

which seems to be the same metric as that obtained from $\mathcal{M}_{(S,S)}$. There is, however, an obstruction against regarding (5.14) and (5.3) as equivalent. In order to identify (5.14) and (5.3) we have to regard t' in (5.14) to be an ordinary time function which is single-valued on the spacetime. As a consequence, the gauge transformation $g^{(\pm)} = e^{\mp\lambda_0 t}$ which we have used to construct a nonsingular metric becomes a large gauge transformation which relates the non-equivalent connections. It would be more sensible to regard t as a single-valued time function and $g^{(\pm)} = e^{\mp\lambda_0 t}$ to be a gauge transformation which is homotopic to the identity. The spacetime with metric (5.14) is then entirely different from the spacetime with metric (5.3) unless $(n_+, m_+) = (n_-, m_-)$. This can be seen by being aware that the spacetime (5.14) is parametrized by (5.4) with t replaced by t'. The spacetime (5.14), however, does not appear in the ordinary ADM formalism because $(t = const.)$-hypersurface necessarily involves timelike region.

Thus it is not straightforward to deal with the $\widetilde{\mathrm{SL}}(2, \mathbf{R}) \times \widetilde{\mathrm{SL}}(2, \mathbf{R})$ gauge theory. In particular, in the case of the remaining sectors (except $\tilde{\mathcal{M}}_T^{(+)} \times \tilde{\mathcal{M}}_T^{(-)}$) we do not even know whether there exist any spacetimes which correspond to a point on each sector. To elucidate the problems on the $\widetilde{\mathrm{SL}}(2, \mathbf{R}) \times \widetilde{\mathrm{SL}}(2, \mathbf{R})$ gauge theory, more extensive analysis is longed for.

We return to the $\mathrm{SO}(2,2)_0$ gauge theory on neglecting the problms explained above. The eight sectors except $\mathcal{M}_{(S,S)}$ give spacetimes in which each torus T^2 is timelike, so they do not correspond to the ordinary ADM formalism. These space-times are, however, solutions of Einstein's equations as is seen from the fact that they are constructed from the 3-dimensional anti-de Sitter space. So we can consider

that each point in $\mathcal{M}'\backslash\mathcal{M}_{(S,S)}$ gives such an "exotic" spacetime [6]. The timelike tori involved in these spacetimes necessarily contain closed timelike curves, which seem to be forbidden by many works[16] to coexist with an ordinary universe which is (at least partially) equipped with a causal structure. The spacetime discussed here is, however, the "nether world" in which all "constant-time" hypersurfaces are timelike, or the spacetime formed by gluing an ordinary universe and the "nether world" using a singularity as a glue. Such spacetimes does not seem to be supressed by [16], and might play an important role in the quantum gravity particularly when we describe the epoch before and during the big bang, as euclidean spacetimes do in the path integral approaches. To see whether this is indeed the case, it would be necessary to investigate the physical adequacy of these spacetimes more rigorously.

We know that the $\mathcal{M}_{(S,S)}$ is in 1 to 2 correspondence with the ADM formalism [12]. Now we investigate the origin of this 1 to 2 correspondence.

We have seen that the 1 to 2 correspondence is originated from the symmetry of $\mathcal{M}_{(S,S)}$ under the seemingly discrete transformation

$$(\alpha_+, \beta_+, \alpha_-, \beta_-) \rightarrow (\alpha_+, \beta_+, -\alpha_-, -\beta_-).$$

This transformation is, in fact, generated by the gauge transformation $(G^{(+)}(\pi), G^{(-)}(\pi)) \equiv (1, \exp(\pi\lambda_0))$ which belongs to the 1-parameter family of transformations:

$$(G^{(+)}(\theta), G^{(-)}(\theta)) \equiv (1, \exp(\theta\lambda_0)). \tag{5.15}$$

By performing on the connection $A^{(\pm)} = \lambda_2 d\varphi_\pm$ in $\mathcal{M}_{(S,S)}$ the gauge transformation $G^{(\pm)}(\theta)$ and a time-dependent gauge transformation $g^{(\pm)} = e^{\mp\lambda_0 t}$, we obtain the $SO(2,2)_0$ connection whose triad part is given by

$$
\begin{aligned}
E_\theta^0 &= dt \\
\begin{pmatrix} E_\theta^1 \\ E_\theta^2 \end{pmatrix} &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -\sin(t+\frac{\theta}{2})\frac{d\varphi_+ + d\varphi_-}{2} \\ -\cos(t+\frac{\theta}{2})\frac{d\varphi_+ - d\varphi_-}{2} \end{pmatrix}.
\end{aligned}
\tag{5.16}
$$

The transformation which lead from (5.2) to (5.16) is the composition of a spatial

rotation and a "time-shift" [9]

$$t \to t + \frac{\theta}{2}.$$

So we can arbitrarily choose the origin of time. If we only consider the region which does not have singularity on the way of time evolution, then this symmetry tells us that we cannot distinguish the universes whose metric is given by (5.3) with the regions of time being $(-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2})$ respectively. We know that there are two types of singularities in the region parametrized by (5.4), which are the lines $t = n\pi$ and $t = (n + \frac{1}{2})\pi$. Each of the above two universes begins with one of these singularities and ends with the other. We can conclude that the 1 to 2 correspondence is originated from the lack of criterion for choosing the origin of time in our prescription to construct spacetimes.

We could explain this 1 to 2 correspondence from the viewpoint of the $SO(2, 2)_0$ holonomy. From one holonomy group, we can construct two different spacetimes, e.g. the spacetimes obtained by identifying the regions $\{T > |X|, Z > |Y|\}$ and $\{T < -|X|, Z > |Y|\}$ using the same holonomy (5.5). What is peculiar to the $\Lambda < 0$ case is that we can obtain the above spacetimes also by identifying the regions $\{T < -|X|, Z > |Y|\}$ and $\{T > |X|, Z > |Y|\}$ using the different (but gauge-equivalent) holonomy

$$\tilde{E}[\alpha] = \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cosh \alpha & \sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{pmatrix}$$

$$\tilde{E}[\beta] = \begin{pmatrix} \cosh v & \sinh v & 0 & 0 \\ \sinh v & \cosh v & 0 & 0 \\ 0 & 0 & \cosh \beta & \sinh \beta \\ 0 & 0 & \sinh \beta & \cosh \beta \end{pmatrix}. \tag{5.17}$$

We can consider this peculiar nature of the holonomy in the anti-de Sitter case to be the origin of 1 to 2 correspondence.

---

[9] We can redard this "time-shift" as a temporal diffeomorphism, provided that a shift of the region of t, e.g. from $(0, \frac{\pi}{2})$ to $(-\frac{\theta}{2}, \frac{\pi - \theta}{2})$, follows. If t runs in the region $(-\infty, \infty)$, they cannot be distinguished. In that case, however, we have to deal with the universe with singularities on the way of time evolution [12].

6 Toward the Quantum Theory

In this section we try to quantize the "unified" phase space $\mathcal{M}'$ in the $\mathrm{SO}(2,2)_0$ gauge theory. We first look into the classical transformation property under the large diffeomorphisms and then we construct the modular invariant quantum theory on a "modified phase space" $\mathcal{M}^\circ$. Finally we investigate the relation to the quantum theory in ref [3] which is related to the quantum ADM formalism.

As we have seen $\mathcal{M}'$ does not have a cotangent bundle structure. The most familiar quantization where quantum states are represented by functions of *coordinates* is, however, defined only when the phase space allows a "real polarization", whose typical example is a cotangent bundle structure. By an artifitial prescription we deform the $\mathcal{M}'$ into $\mathcal{M}^\circ$ which is a cotangent bundle on a torus.

6.1 Modular transformations

First we look into the behaviour of our new canonical variables under large diffeomorphisms, in particular the inversion:

$$I : \ (\alpha, \beta) \longrightarrow -(\alpha, \beta) \qquad (or \quad (x, y) \to -(x, y)), \tag{6.1}$$

which induces the following *simultaneous* transformations:

$$I : \ (\theta_\pm, r_\pm) \longrightarrow (\theta_\pm + \pi, r_\pm). \tag{6.2}$$

$\mathcal{M}'$ does not have a cotangent bundle structure even after imposing this symmetry. If we perform the following *artificial* prescription, however, the resulting phase space $\mathcal{M}^\circ$ acquires a cotangent bundle structure $\mathcal{M}^\circ = \mathbf{T}^*\mathcal{B}$ with the base space $\mathcal{B} \approx T^2$: i) First we get rid of the 2π-periodicity in $\rho_\pm$ and $\sigma_\pm$ which parametrize $\mathcal{M}_T^{(\pm)}$ and make $\mathcal{M}_T^{(\pm)}$ homeomorphic to $\mathbf{R}^2\backslash\{(0,0)\}$. ii) We assume that $(\theta_+, \theta_- + \pi)$ can be distinguished from (θ_+, θ_-) *even when either r_+ or r_- is negative*. This involves the assumption that the $\mathcal{M}_{(S,S)}$ is not in 1 to 2 correspondence but equivalent with the ADM phase space.

The "modified" phase space $\mathcal{M}^\circ$ constructed as above has a symplectic potential

$$\Theta = L(r_+\delta\theta_+ - r_-\delta\theta_-) \in T^*\mathcal{B} \quad (\delta\Theta = \Omega) \tag{6.3}$$

with the base space $\mathcal{B}$ being parametrized by

$$(\theta_+, \theta_-) \sim (\theta_+ + \pi, \theta_- + \pi) \sim (\theta_+ + 2\pi, \theta_-). \tag{6.4}$$

What is the meaning of the "modified" phase space $\mathcal{M}^\circ$? The phase space of general relativity should be composed of equivalent classes of solutions of Einstein's equations under the diffeomorphisms. The double covering of $\mathcal{M}_{(S,S)}$ is equivalent to the phase space of the ADM formalism. In CSG, however, the phase space is expected to be extended compared to the ADM phase space because CSG can contain singularity where the spatial metric collapses. We can regard $\mathcal{M}^\circ$ to be the phase space of the model which take such effect of CSG into account to some extent. To obtain the *true* phase space of CSG, we have to start with the $\widetilde{\mathrm{SL}}(2, \mathbf{R}) \times \widetilde{\mathrm{SL}}(2, \mathbf{R})$ gauge theory and complete the investigation made in the last section.

Next we investigate the behaviour under the modular group $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. Transformations of the classical variables under the two elementary modular transformations:

$$S : (\alpha, \beta) \rightarrow (-\beta, \alpha), \quad T : (\alpha, \beta) \rightarrow (\alpha + \beta, \beta),$$

prove to be given by the following simultaneous transformations

$$S \; : \; (\theta_\pm, r_\pm) \rightarrow (\theta_\pm + \frac{\pi}{2}, r_\pm), \tag{6.5}$$

$$T \; : \; (\theta_\pm, r_\pm) \rightarrow \left(\frac{1}{i} \ln \left\{ \frac{e^{i\theta_\pm} + \sin\theta_\pm}{\sqrt{1 + \sin 2\theta_\pm + \sin^2\theta_\pm}} \right\}, (1 + \sin 2\theta_\pm + \sin^2\theta_\pm)r_\pm \right).$$

We can show that these transformations preserve the symplectic structure of $\mathcal{M}'$ and the cotangent bundle structure of $\mathcal{M}^\circ$. We have only to show that the symplectic potential Θ is also a well-defined section of $T^*(\mathcal{B}/\Gamma)$, i.e. that the values of Θ before and after the transformation coincide. As for S, it is straightforward. Invariance under T is demonstrated as: [10]

$$
\begin{aligned}
T^*\Theta \; &= \; L[T(r_+)\delta(T(\theta_+)) - T(r_-)\delta(T(\theta_-))] \\
&= \; L[(1 + \sin^2\theta_+ + \sin 2\theta_+)r_+\delta\left\{ \frac{1}{i} \ln \frac{e^{i\theta_+ + \sin\theta_+}}{\sqrt{1 + \sin^2\theta_+ + \sin 2\theta_+}} \right\} - \{(+) \leftrightarrow (-)\}] \\
&= \; L(r_+\delta\theta_+ - r_-\delta\theta_-) = \Theta. \tag{6.6}
\end{aligned}
$$

---

[10]T^* here does not denote a cotangent bundle but denotes a pull-back of a form on $\mathcal{M}^\circ$ under the Dehn twist T.

We therefore expect that under the assumption made above a consistent quantum theory can be defined on the "fundamental region" $\mathcal{B}/\Gamma$.

6.2 Quantum Theory on the Modified Phase Space

If we use the cotangent bundle structure $\mathcal{M}^\circ = T^*\mathcal{B}$, we can construct a representation where the quantum states are functions of (θ_+, θ_-). In the quantum theory the canonical variables $(\theta_\pm, r_\pm)$ are promoted to the basic operators which satisfy the canonical commutation relations derived from the symplectic structure (3.11):

$$[\hat{\theta}_\pm, \hat{r}_\pm] = \pm i \frac{1}{L} \quad , \quad \textit{zero otherwise.} \tag{6.7}$$

It is probable that the action of $\hat{\theta}_\pm$ on the wavefunction χ is given by multiplication

$$\hat{\theta}_\pm \chi(\theta_+, \theta_-) = \theta_\pm \cdot \chi(\theta_+, \theta_-) \quad . \tag{6.8}$$

To determine the action of $\hat{r}_\pm$, however, we have to know the integration measure or the inner product. It would be natural that the inner product is invariant under the modular group $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. If we require the modular invariance of the squared modulus $\overline{\chi}\chi$ of the wave function, one of the candidates is given by

$$< \chi_1 | \chi_2 >= \int \int \frac{d\theta_+ d\theta_-}{\sin^2(\theta_+ - \theta_-)} \overline{\chi_1(\theta_+, \theta_-)} \chi_2(\theta_+, \theta_-). \tag{6.9}$$

Modular invariance of the integration measure $\frac{d\theta_+ d\theta_-}{\sin^2(\theta_+ - \theta_-)}$ can be demonstrated by a direct calculation using (6.5).

If we require the action of $\hat{r}_\pm$ to be self-adjoint with respect to the inner product (6.9),we find

$$\begin{aligned} \hat{r}_\pm \chi(\theta_+, \theta_-) &= \frac{1}{L}\left[\mp i \frac{\partial}{\partial \theta_\pm} + i \cot(\theta_+ - \theta_-)\right]\chi(\theta_+, \theta_-) \\ &= \mp \frac{i}{L}\sin(\theta_+ - \theta_-)\frac{\partial}{\partial \theta_\pm}\left(\frac{1}{\sin(\theta_+ - \theta_-)}\chi(\theta_+, \theta_-)\right). \end{aligned} \tag{6.10}$$

To determine the modular transformation of the quantum operators, we have to consider the issue of operator ordering seriously. Transformation of $\hat{\theta}_\pm$ under S, T and transformation of $\hat{r}_\pm$ under S are obtained by directly promoting the transformation (6.5) to the operator relation. Transformation of $\hat{r}_\pm$ under T, however,

involves the operators $r_\pm$ and $\theta_\pm$ which do not commute and so we have to determine the operator ordering.

If, for example, we require the self-adjointness of $\hat{r}_\pm$ to be preserved under the T-transformation, the transformation is

$$
\begin{aligned}
T : \hat{r}_\pm \quad &\rightarrow \quad (1 + \sin 2\hat{\theta}_\pm + \sin^2 \hat{\theta}_\pm)^{1/2} \hat{r}_\pm (1 + \sin 2\hat{\theta}_\pm + \sin^2 \hat{\theta}_\pm)^{1/2} \\
&= \quad \mp \frac{i}{L} \sin(T(\theta_+) - T(\theta_-)) \frac{\partial}{\partial T(\theta_\pm)} \left(\frac{1}{\sin(T(\theta_+) - T(\theta_-))} \right).
\end{aligned}
\quad (6.11)
$$

For the operators to transform in these ways, the wave function $\chi(\theta_+, \theta_-)$ must be invariant, up to a constant phase factor, under the modular transformations.

Since the quantum theory constructed as above is defined on $\mathcal{M}^\circ$ which is larger than the ADM phase space, we may find a process which is not expected by quantizing the ADM formalism. In our quantum theory, momentum eigenstates would play an important role because each sector is identified by the signature of (r_+, r_-).

6.3 Quantum Relation between New and Old Parametrizations

Here let us investigate the relation between two representations in which wave functions are functions of old parameters (α, β) and functions of new parameters (θ_+, θ_-), respectively. We expect that such relation is given by a sort of integral transformation.

In ref .[3], Carlip derived the integral transformation from quantum ADM formalism to quantum CGG by extracting the eigenfunction of modulus operator $\hat{m}$ in the quantum CSG and by using it as the kernel. In our case, however, it is difficult to perform such prescription because the relation between old parameters (α, β, u, v) and new parameters $(\theta_\pm, r_\pm)$ is non-polynomial as is shown by (3.9). So we use other method which invokes the geometric quantization [13].

We breifly explain the "orthgonal projection"[13] by using the situation where a phase space $\mathcal{M}$ admits two transverse real polarizations P and P'. The base spaces $Q = \mathcal{M}/P$ and $Q' = \mathcal{M}/P'$ are parametrized by the coordinates q^i and q'^i respectively. We denote the conjugate momenta of q^i and q'^i by p_i and p'_i respectively.

Suppose that the canonical transformation is written as

$$p_i dq^i = p_i' dq'^i + dS(q^i, q'^i), \tag{6.12}$$

and that the measures of inner product in the representations based on Q and Q' are given by $\mu(q)d^n q$ and $\mu'(q')d^n q'$ respectively. Then the integral transformation from the representation based on Q' to that based on Q is given by

$$\psi(q^i) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{Q'} \left(\frac{\mu'}{\mu}\right)^{1/2} d^n q' \sqrt{\det\left(\frac{\partial^2 S}{\partial q^j \partial q'^k}\right)} e^{iS} \psi(q'^i). \tag{6.13}$$

Let us apply this formula to the double covering of $\mathcal{M}_{(S,S)}$. We replace q^i and q'^i by $(\theta_+, \theta_-) \in \mathcal{B}$ and $(\alpha, \beta) \in \mathbf{R}^2/\mathbf{Z}_2$ respectively. The canonical transformation between these variables is given by (4.11). If we substitute these into (6.13), we obtain the desired integral transformation

$$\chi(\theta_+, \theta_-) = \int d\alpha d\beta \frac{\sqrt{-2LV}}{\pi} e^{iV} \chi(\alpha, \beta). \tag{6.14}$$

Owing to the modular invariance of V, $\chi(\theta_+, \theta_-)$ becomes modular invariant if we require $\chi(\alpha, \beta)$ to be modular invariant (up to a constant phase factor). To justify this integral transformation, however, more extensive investigation are needed as to, for example, the relations between the operators $(\hat{\theta}_\pm, \hat{r}_\pm)$ and $(\hat{\alpha}, \hat{\beta}, \hat{u}, \hat{v})$. This is expected to be complicated and is left to the future investigation.

Finally we shall make a digression. We could formally apply this "orthogonal projection" method to the derivation of the quantum relation between the ADM formalism and CSG. We replace q^i and q'^i by (α, β) and (m_1, m_2) respectively. Using the canonical transformation (4.5) we find

$$\chi(\alpha, \beta) = \int \frac{d^2 m}{m_2{}^2} \frac{|\beta - m\alpha|}{\pi \tilde{\tau} \sqrt{m_2}} e^{-\frac{i}{m_2 \tilde{\tau}}|\beta - m\alpha|^2} \chi(m_1, m_2), \tag{6.15}$$

where $\tilde{\tau} \equiv \frac{1}{L} \cot t$. This expression is different from the integral transformation

$$\chi(\alpha, \beta) = \int \frac{d^2 m}{m_2{}^2} \frac{\beta - m\alpha}{\pi \tilde{\tau} \sqrt{m_2}} e^{-\frac{i}{m_2 \tilde{\tau}}|\beta - m\alpha|^2} \chi(m) \tag{6.16}$$

which is derived by Carlip [3] by a phase factor $\exp\{i\arg(\beta - m\alpha)\}$ in the kernel. This is probably because we have applied the "orthogonal projection" method naively to the time-dependent canonical transformation (4.5). It would be no wonder that a modification is required in the case of a time-dependent canonical transformation.

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7 Discussion

In this paper we have investigated Chern-Simons formulation of anti-de Sitter gravity on $\mathbf{R} \times T^2$ with an emphasis on the properties of the whole phase space. In particular, we have shown that the nine sectors which appeared in ref.[12] are in fact not disconnected but are mutually connected to form the "unified" phase space $\mathcal{M}'$, which is a direct product of two copies of a non-Hausdorff manifold, plus a set with nonzero codimensions. We have also seen that each point on $\mathcal{M}'$ corresponds to a spacetime (or spacetimes) which is a solution of Einstein's equations with a negative cosmological constant. In order to quantize this theory in a conventional fashion, we have made an artifitial prescription to modify $\mathcal{M}'$. $\mathcal{M}^\circ$ obtained in this way enjoys a cotangent bundle structure which is preserved under the modular transformations. This property is convenient to the one who want to construct a modular invariant quantum theory. Though somewhat formally, we have also given the relation between our new quantum theory and the quantum theory which was given in [12] and which is closely related to the spinor representation [3] of the ADM formalism.

While we have investigated CSG on $\mathbf{R} \times T^2$ considerably extensively, there remain many issues to be resolved in order to complete the analysis. We will list some of these issues.

In giving the spacetime interpretation to $\mathcal{M}'$, which is obtained by regarding the gauge group as $\mathrm{SO}(2,1)_0 \times \mathrm{SO}(2,1)_0$, we have seen that infinitely many spacetimes correspond to each point on the (T, Φ)- or the (Φ, T)-sectors. Thus we expect $\tilde{\mathcal{M}}_U^{(+)} \times \tilde{\mathcal{M}}_U^{(-)}$, which is obtained by choosing $\widetilde{\mathrm{SL}}(2, \mathbf{R}) \times \widetilde{\mathrm{SL}}(2, \mathbf{R})$ as the gauge group, to be more suitable to the spacetime interpretation. Relating spacetimes to the all points on $\tilde{\mathcal{M}}_U^{(+)} \times \tilde{\mathcal{M}}_U^{(-)}$, however, requires a considerable exertion. Moreover, the choice as to whether we identify the different points on $\tilde{\mathcal{M}}_U^{(+)} \times \tilde{\mathcal{M}}_U^{(-)}$ which give the same spacetime or not changes the structure of the "physical phase space" drastically.[11] After we construct the "true" phase space in CSG, we have to quantize this phase space using the geometric quantization scheme. Though the quantum theory which we have given in §6 is constructed on the modified phase space, it is

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[11] We can see a similar example in ref.[17], which deals with the de Sitter case.

based on the method of the geometric quantization and so we can probably extend the prescription developed in §6 to the complete quantization of the "true" phase space. To accomplish this task it is necessary to find out the complete quantum relation between the old and the new parametrizations.

We should note that the spacetimes we have given are not the unique ones constructed from the points in $\mathcal{M}'$. It is because the gauge group $SO(2,2)_0$ (or $\widetilde{SL}(2,\mathbf{R}) \times \widetilde{SL}(2,\mathbf{R})$) is in fact larger than the semi-direct product of the 2+1 dimensional local Lorentz group and the group of diffeomorphisms [6][7]. For illustration, we consider $\mathcal{M}_{(S,S)}$. By choosing time dependent gauge transformations other than that giving the spacetime (5.3), we can construct various spacetimes. There are for example Louko-Marolf-type universe [6] and Unruh-Newbury-type universe [7] in which timelike tori appear. Though these spacetimes coincide with one another in the region where the ADM is well-defined ($T > |X|$, $Z > |Y|$), their behaviors in the other region vary considerably by the choice of gauge. At present there seems to be no criterion for choosing the most relevant gauge.

In §5 we have investigated the origin of the 1 to 2 correspondence with the ADM formalism. In the de Sitter case, there exists 1 to ∞ correspondence[17], whose origin also have to be elucidated. We consider that this 1 to 2 correspondence is

closely related to the fact that the $SO(3,1)$ gauge group is in fact larger than the semi-direct product of the $(2+1)$-local Lorentz group and the diffeomorphism group, in particular when the triad is degenerate.

To extract instructions on the $(3+1)$-dimensional quantum gravity, it is necessary to compare the reduced phase space method which has been discussed in this paper to Dirac's quantization method[18]. Witten has applied this Dirac's quantization in the de Sitter case with the help of geometric quantization [19]. It is worth investigating whether Witten's prescription can be extended to the anti-de Sitter case.

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